Review for Poisson equation and Laplace equation.
(For concrete computation, refer to Evans)
Laplace equation $\Delta u = 0$ / $-\Delta u = 0$
Poisson equation $-\Delta u = f$.

To find the radial solutions $v(r)$, $r = |x|$ for $\Delta v = 0$.

Consider $\begin{cases} v_{x_i} = v'(r) \cdot \frac{x_i}{r} \\ v_{x_i x_i} = v''(r) \cdot \frac{x_i^2}{r^2} + v'(r) \cdot \left(\frac{1}{r} - \frac{x_i}{r^2} \cdot \frac{x_i}{r} \right) \end{cases}$ $\quad x \in \mathbb{R}^d$

$\Rightarrow v''(r) + v'(r) \cdot \frac{d-1}{r} = 0$

$\Rightarrow v(r) = \begin{cases} -\frac{1}{2\pi} \log|x|, & d = 2 \\ \frac{1}{d(d-2)\alpha(d)} \frac{1}{|x|^{d-2}}, & d \geq 3 \end{cases} \hat{=} \Phi(x)$

the volume of unit ball in $\mathbb{R}^d$.

So why we choose the constant $-\frac{1}{2\pi}$, $\frac{1}{d(d-2)\alpha(d)}$?
The answer is this meets the requirement for solving Poisson eq, though it's designed to solve Laplace equation.

- If $f \in C_c^2(\mathbb{R}^n)$, $\Phi * f$ solves $-\Delta u = f$ with $u \in C^2(\mathbb{R}^n)$.

The proof is done by $(\Phi * f)_{x_i x_j} = \Phi * f_{x_i x_j}$, leverage $f \in C_c^2(\mathbb{R}^n)$.

Meanwhile, we have: $-\Delta \Phi = \delta(x)$, which means $\Phi(x)$ is the fundamental solution.

- Meanvalue formula: consider now an open set $U \subset \mathbb{R}^n$ and suppose u is a harmonic function within U. If $u \in C^2(U)$.

Then $u(x) = ⨍_{\partial B(x,r)} u(y)\, dS(y) = ⨍_{B(x,r)} u(y)\, dy$.

Proof: Consider $\varphi(r) = ⨍_{\partial B(x,r)} u(y)\,dS(y) = ⨍_{\partial B(0,1)} u(x+rz)\,dS(z)$

$$\varphi'(r) = ⨍_{\partial B(0,1)} Du(x+rz)\cdot z\,dS(z) \Rightarrow C^1.$$

$$= ⨍_{\partial B(x,r)} Du(y)\frac{y-x}{r}\,dS(y) = \frac{r}{n}⨍_{B(x,r)} \Delta u(y)\,dy = 0$$

$$\Rightarrow u(x) = \lim_{r\to 0^+}\varphi(r) = \varphi(r)$$

Meanwhile. $⨍_{B(x,r)} u(y)\,dy = \frac{1}{\alpha(n)r^n}\int_0^r \int_{\partial B(x,s)} u(y)\,dS(y)\,ds$

$$= \frac{1}{\alpha(n)r^n}\int_0^r u(x)\cdot n\alpha(n)s^{n-1}\,ds = u(x).$$

• (Converse) If $u \in C^2(U)$ satisfies $u(x) = ⨍_{\partial B(x,r)} u\,dS$ for each ball $B(x,r) \subset U$, then u is harmonic.

Proof: We have $0 = \varphi'(r) = \frac{r}{n}⨍_{B(x,r)} \Delta u(y)\,dy \neq 0$ ... ☒
($\exists B(x,r) \subset U$, s.t $\Delta u < 0$ or $\Delta u > 0$)

• (Strong maximum principle) Suppose $u \in C^2(U) \cap C(\bar{U})$ is harmonic within U

① $\max_{\bar{U}} u = \max_{\partial U} u$

② If U is connected and $\exists x_0 \in U$, s.t $u(x_0) = \max_{\bar{U}} u \Rightarrow u$ is a constant within U

Proof: Suppose there exists a point $x_0 \in U$, s.t $u(x_0) = \max_{\bar{U}} u$.
From mean-value theorem. $\forall x \in B(x_0,r) \subset U$, $u(x) = u(x_0)$
Since $u \in C(\bar{U})$, we can see that: in U, the maximum point's set is both open and closed.
So ② is obviously, from the property of connected sets
As for ①. you can extending the ball $B(x,r)$. unless it reaches the boundary of u, if there exists $x = \arg\max_{\bar{U}} u \in U$.

• (Smoothness): If $u \in C(U)$ ($\subset \mathbb{R}^d$) satisfies the mean-value property for each ball $B(x,r) \subset U$, then $u \in C^\infty(U)$.

Pf: Consider the $\eta(x) = \begin{cases} C e^{\frac{1}{|x|^2-1}}, & |x|<1 \\ 0, & |x| \geq 1. \end{cases}$ C, s.t. $\int \eta(x)dx = 1$

$\eta_\varepsilon(x) = \varepsilon^{-d} \eta(\frac{x}{\varepsilon})$, $\begin{cases} \eta_\varepsilon * u \triangleq u_\varepsilon \in C^\infty(U_\varepsilon), \ U_\varepsilon = \{x \in U \mid dist(x, \partial U) \geq \varepsilon\} \\ u_\varepsilon \to u \text{ a.e. as } \varepsilon \to 0 \end{cases}$

Obviously, u is locally integrable.

We only need to prove $u_\varepsilon \equiv u$ in U_ε

$$u_\varepsilon = \int \eta_\varepsilon(x-y)u(y)dy = \int_{B(x,\varepsilon)} \eta_\varepsilon(x-y)u(y)dy$$

$$= \varepsilon^{-d}\int_{B(x,\varepsilon)} \eta(\tfrac{|x-y|}{\varepsilon})u(y)dy$$

$$= \varepsilon^{-d}\int_0^\varepsilon \eta(\tfrac{r}{\varepsilon})\int_{\partial B(x,r)} u\, dS\, dr$$

$$= \varepsilon^{-d}\int_0^\varepsilon \eta(\tfrac{r}{\varepsilon})\, d\alpha(d)\, r^{d-1} u(x)\, dr$$

$$= u(x)\int_{B(0,\varepsilon)} \eta_\varepsilon\, dy = u(x)$$

Good

• (Estimate on derivative). If u is harmonic in U, then

$$|D^\alpha u(x_0)| \leq \frac{C_{|\alpha|}}{r^{n+|\alpha|}} \|u\|_{L^1(B(x_0,r))} \quad \text{for each ball } B(x_0,r) \subset U$$

related to $|\alpha|$, n,

Proof: Induction on $|\alpha| \triangleq k$ is enough.

First, from above $u \in C^\infty(U)$. Hence. u_{x_i} is still a solution of $\Delta u = 0$

Assume that $B(x,r) \subset U \subset \mathbb{R}^n$.

① And $|u_{x_i}| = \left| \fint_{B(x,\frac{r}{2})} u_{x_i}(y)dy \right|$

$$= \left|\frac{2n}{r}\fint_{\partial B(x,\frac{r}{2})} u(y)\cdot \nu_i dS(y)\right|.$$

normal vector corresponding to x_i

$$\leq \frac{2^n \cdot n\alpha(n)(\frac{r}{2})^{n-1}}{\alpha(n)r^n} \cdot \|u\|_{L^\infty(\partial B(x,\frac{r}{2}))} = \frac{2n}{r}\|u\|_{L^\infty(\partial B(x,\frac{r}{2}))}$$

Notice that: $\|u\|_{L^\infty(\partial B(x,\frac{r}{2}))} \le \frac{\|u\|_{L^1(B(x,r))}}{\alpha(n)(\frac{r}{2})^n}$

This finishes the proof of $|\alpha|=1$

② $|\alpha|=k \to k+1=|\beta|$

Notice that $D^\beta u = (D^\alpha u)_{x_i}$ for some i.

$$|D^\beta u| = |(D^\alpha u)_{x_i}| = \left| ⨍_{B(x,\frac{r}{k})} (D^\alpha u)_{x_i} dx \right|$$

$$= \left| \frac{1}{\alpha(n)(\frac{r}{k})^n} \int_{\partial B(x,\frac{r}{k})} D^\alpha u \cdot \nu_i \, dx \right|$$

$$\le \frac{nk}{r} \|D^\alpha u\|_{L^\infty(\partial B(x,\frac{r}{k}))}$$

Meanwhile, $\|D^\alpha u\|_{L^\infty(\partial B(x,\frac{r}{k}))} \le \frac{C_{|\alpha|}}{(\frac{k-1}{k}r)^{n+k}} \|u\|_{L^1(B(x,r))}$

So, we have:

(Liouville): $u: \mathbb{R}^n \to \mathbb{R}$ harmonic and bounded, then u is a constant.

Proof: $|D^\alpha u| \le \frac{C_1}{r^{n+1}} \|u\|_{L^1(B(x,r))}$

$$\le \frac{C_1 \alpha(n)}{r} \|u\|_{L^\infty} \quad (r \to \infty) \to 0 \quad \text{for all } |\alpha|=1.$$

(Representation formula): Let $f \in C_c^2(\mathbb{R}^n)$, $n \ge 3$, Then any bounded solution of $-\Delta u = f$ in $\mathbb{R}^n$, has the form:

$$u(x) = \int \Phi(x-y) f(y) dy + C.$$

Proof: Since $\Phi(x) \to 0$ as $|x| \to +\infty$ when $n \ge 3$

So $\Phi * f$ gives a bounded solution of $-\Delta u = f$, then by Liouville thm, ... ☑

Here, we give a last glance at harmonic functions.

·(Harnack ineq): For each connected open set $V \subset\subset U$ ($V \subset \bar{V} \subset U$), there exists a positive constant C, depending only on V, s.t.

$$\sup_V u \le C \inf_V u \text{ for all harmonic non-negative functions in } U$$

Proof: Lest $r = \frac{1}{4}\operatorname{dist}(\bar{V}, \partial U)$.

Consider: $\forall |x-y| < r$, $B(x,2r) \supset B(y,r)$

$$\Rightarrow u(x) = \frac{1}{\alpha(n)(2r)^n} \int_{B(x,2r)} u(y)\,dy \ge \frac{1}{\alpha(n)(2r)^n} \int_{B(y,r)} u(z)\,dz \ge \frac{u(y)}{2^n}$$

V is connected + $\bar{V}$ is cpt $\Rightarrow$ $\bar{V}$ can be covered by chain of finite many balls $\{B_i\}_{i=1}^{N}$, B_i's radius are all $\frac{r}{2}$ with $B_i \cap B_{i-1} \ne \emptyset$

$$\Rightarrow \forall x, y \in V, \ u(x) \ge \frac{u(y)}{2^{n(N+1)}}$$ ☑

Then, we take a look at Green's function for Poisson equation with boundary condition.
(Assume $U \subset \mathbb{R}^n$ open, ∂U is C^1).

$$\begin{cases} -\Delta u = f, & x \in U \\ u = g & \text{on } \partial U \end{cases} \quad ; f, g \text{ are continous.}$$

For each $y \in \mathbb{R}^n$, let $\phi^x(y)$ solves $\begin{cases} \Delta \phi^x(y) = 0, & U \\ \phi^x(y)|_{\partial U} = \Phi(y-x) \end{cases}$ (fix $x \in U$)

Let $G(x,y) = \Phi(y-x) - \phi^x(y)$, $(x, y \in U, x \ne y)$ $= G(y,x)$ (a little trick)

Then fix x, $G(x,\cdot)$ solves $\begin{cases} -\Delta G = \delta_x & \text{in } U \\ G = 0 & \text{on } \partial U \end{cases}$

And the representation formula using Green's function is

If $u \in C^2(\bar{U})$ solves this problem,

$$u(x) = -\int_{\partial U} \frac{\partial G}{\partial \nu}(x,y) g(y) \, dS(y) + \int_U G(x,y) f(y) \, dy$$

We now use this to solve

I): Poisson (In fact, Laplace, since $f \equiv 0$ here) in $\mathbb{R}^n_+ = \{(x_1, \cdots, x_n) \in \mathbb{R}^n,$ $x_n > 0\}$, then $\partial \mathbb{R}^n_+ = \{(x_1, \cdots, x_n) \in \mathbb{R}^n \mid x_n = 0\}$

For $x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n_+$, Let $\tilde{x} = (x_1, x_2, \cdots, -x_n)$.
Then $\phi^x(y) = \Phi(y - \tilde{x})$ satisfies the requirement

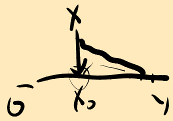
So $G(x,y) = \Phi(y-x) - \Phi(y-\tilde{x})$. (Assume $n \geq 3$ here)

$$\frac{\partial G}{\partial \nu} G(x,y) = -(\Phi_{y_n}(y-x) - \Phi_{y_n}(y-\tilde{x}))$$
$$= -\frac{2x_n}{n\alpha(n)} \frac{1}{|x-y|^n}$$

$\Rightarrow u(x) = \frac{2x_n}{n\alpha(n)} \int_{\partial \mathbb{R}^n_+} \frac{g(y)}{|x-y|^n} \, dS(y)$ $\overset{\Delta}{=} y$, since $\partial \mathbb{R}^n_+ \cong \mathbb{R}^{n-1}$ solves $\begin{cases} \Delta u = 0, & \mathbb{R}^n_+ \\ u|_{\partial \mathbb{R}^n_+} = g \end{cases}$

To prove $u(x)$ indeed solve the equation.

Just notice that: the Poisson Kernel $K(x,y) = \frac{2x_n}{n\alpha(n)|x-y|^n}$ satisfies $\int_{\partial \mathbb{R}^n_+} K(x,y) dy = 1$.

$|x-y|$ 

When $x \to x^0 \in \partial\mathbb{R}^n_+$, $K(x,y)$ forms an approximation identity. (非负+连续)

① $\forall x \in \mathbb{R}^n_+$, $\int K(x,y)\,dy = 1$.

② $\lim\limits_{x\to x^0} \int_{|y-x^0|\geq\varepsilon} K(x,y)\,dy = 0$ (When $x^0 \to x$, $|y-x| \geq \frac{1}{2}|y-x^0|$)

Hence, we can finish the proof easily.

(Require $g \in C(\partial\mathbb{R}^n_+) \cap L^\infty(\partial\mathbb{R}^n_+)$, for approximation identity)

You may ask how to prove u is harmonic inside $\mathbb{R}^n_+$.

Notice that: $y \to G(x,y)$ is harmonic except $x=y$

$\Downarrow$

$x \to G(x,y)$ is harmonic except $x=y$

$\Rightarrow x \to -\frac{\partial G}{\partial y_n}(x,y) = K(x,y)$ is harmonic for $x \in \mathbb{R}^n_+$, $y \in \partial\mathbb{R}^n_+$.

And $\int K(x,y)\,dy = 1$ with $g \in L^\infty(\partial\mathbb{R}^n_+) \Rightarrow u$ is bounded.

And $x \to K(x,y)$ is smooth for $x \neq y$, with $\Delta u(x) = \int_{\partial\mathbb{R}^n_+} \Delta_x K(x,y)\,g(y)\,dy = 0$

...

II) Poisson eq for a ball $B(0,1)$.

Similarly, we define for $x \in B(0,1)$ its dual point by $\tilde{x} = \frac{x}{|x|^2}$

Consider y on $\partial B(0,1)$, we have, $|x-y|^2 = x^2 - 2xy + y^2$

$(|x||y-\tilde{x}|)^2 = x^2 + 1 - 2xy$

Hence, $\phi^x(y) = \phi(|x||y-\tilde{x}|)$

$\Rightarrow G(x,y) = \Phi(y-x) - \Phi(|x||y-\tilde{x}|)$

$$G_{y_i} = \frac{1}{n\alpha(n)} \frac{x_i - y_i}{|x-y|^n} - \frac{1}{n\alpha(n)} \frac{y_i|x|^2 - x_i}{(|x||y-\tilde{x}|^n)} \cdots$$

$$\Rightarrow K(x,y) = \frac{1-|x|^2}{n\alpha(n)} \frac{1}{|x-y|^n}$$

cpt!, hence L^∞

Similarly, if $g \in C(\partial B(0,1)) \Rightarrow u \in C^\infty(B(0,r)), \Delta u = 0$, with the initial condition satisfied.

Now, we give an interesting method to find out the existence of the solution $\begin{cases} \Delta u = 0, & U \\ u = \varphi, & \partial U. \end{cases}$ for U is open.

Def. we say a function f is subharmonic in U if

(i) $f \in C^2$, $\Delta f \geq 0$ in U

(ii) $f \in C(U)$, for any ball $B \subset U$, and any harmonic functions g, if $g|_{\partial B} \geq f|_{\partial B} \Rightarrow g \geq f$ in B.

Def: Harmonic lifting:

Consider a subharmonic function w defined on U, for a ball $B \subset U$ we define. $w_1 = \begin{cases} w, & U \setminus B \\ \Delta w_1 = 0, & B. \end{cases}$ with the same boundary condition.

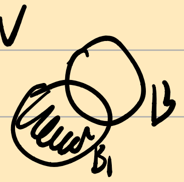
(From def, $w_1 \geq w$)

(Notice that: w_1 exists since on $\partial(U \setminus B)$, w is cts.).

Then, we have: w_1 is subharmonic as well.

Proof: Consider for any ball $B_1 \subset U$, any harmonic function v

If $v \geq w_1|_{\partial B_1} \geq w|_{\partial B_1} \Rightarrow v \geq w$ in $B_1 \Rightarrow v \geq w_1$ in $B_1 \setminus B$



In $B_1 \cap B$, notice that: On $\partial(B_1 \cap B) = (\partial B_1 \cap \bar{B}) \cup (\partial B \cap \bar{B}_1)$

we have $v \geq w_1$ And since they are both harmonic, ...

Now, for $\varphi \in C(\partial U)$, define:

$A\varphi = \{$all subharmonic functions g with $g \le \varphi$ on $\partial U\}$

$$U\varphi(x) = \sup_{g \in A\varphi} g(x)$$

① $U\varphi(x)$ is harmonic. in U

Pf: Only need to prove that: for each $B(x,r) \subset U$, $U\varphi(x)$ is harmonic.

there exists subharmonic functions $\{w_n(x)\}_{n\ge 1}^{\infty}$, s.t $w_n(x) \nearrow U\varphi(x)$. $\subset A\varphi$ as well

consider the harmonic lifting of $w_n(x)$ in $B(x,r) \triangleq B$, $\triangleq \{\widetilde{w}_n(x)\}$

Then $\widetilde{w}_n(x) \nearrow U\varphi(x)$ with $\widetilde{w}_n(x)$ harmonic in B.

Notice that: $\left\{\begin{array}{l}\widetilde{w}_n(x) \text{ is uniformly bounded} \\ \nabla \widetilde{w}_n(x) \text{ is bdd} \Rightarrow \text{equi-cts}\end{array}\right.$ $\xrightarrow{\text{由}A\varphi\text{的定义}} \Rightarrow \exists\, w_{n_k} \rightrightarrows w$ in B.

$\Uparrow$ 导数估计 for harmonic functions.

$\Rightarrow w \in C(U)$ with w is harmonic. with $w(x) \ge U\varphi(x)$

(Notice that: $w_n(y) = \fint_{\partial B(y,r)} w_n(y)\, dS(y)$, $w_n \rightrightarrows w$, we have:

$$w(y) = \fint_{\partial B(y,r)} w(y)\, dS(y)$$

Since $w(y) = \lim_{n\to\infty} w_n(y)$, $\fint_{\partial B(y,r)} w(y)\, dS(y) \ge \liminf_{n\to\infty} \fint_{\partial B(y,r)} w_n(y)\, dS(y)$)

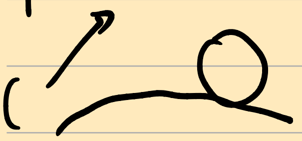
Now we prove that $U\varphi(x') = w(x')$ for all $x' \subset B(x,r)$.

Similarly, at x', there exists subharmonic $v_n \subset A\varphi \nearrow U\varphi(x')$.

W.L.O.G. we assume that $v_n \geq w_n$, ($v_n = \max(v_n, w_n)$)
$\Rightarrow \exists$ harmonic v, s.t $v \geq w$ with $v(x') = u_\varphi(x')$
(v is harmonic in B).
And $w - v$ is harmonic in B with. maximum is achieved in interior
$\Rightarrow w \equiv v$ in $B \Rightarrow \square$. $\boxtimes$

However, to satisfy the boundary condition. there must be given another condition.

(Barrier condition): For each $x_0 \in \partial U$, there exists subharmonic functions Q_φ, s.t $Q_\varphi(x_0) = 0$, $Q_\varphi(x) \leq 0$ when $x \in \partial U \setminus \{x_0\}$

( ① external ball
② C^2-bdry
is sufficient for this)

If the barrier condition is satisfied, then $\lim_{x \to x_0 \in \partial U} u_\varphi = \varphi(x_0)$
(设φ在∂U上有上界为M
Proof: Consider the barrier functions $Q_\varphi(x)$ for x_0.
$\exists \delta > 0$, s.t $|\varphi(x) - \varphi(x_0)| \leq \varepsilon$ for all $|x - x_0| \leq \delta$, $x \in \partial U$.
And there exists $K > 0$, s.t $|K Q_\varphi(x)| \geq 2M$. for $x \in \partial U$, $|x - x_0| > \delta$.

① $\varphi(x_0) - \varepsilon + K Q_\varphi(x)$ is subharmonic with.
in $\partial U \cap B_\delta(x_0)$, $\leq \varphi(x)$
in $\partial U \setminus B_\delta(x_0)$ $\leq \varphi(x)$
$\Rightarrow \in A_\varphi \Rightarrow \leq u_\varphi$.

② $\varphi(x_0) + \varepsilon - K Q_\varphi(x)$ is superharmonic
in $\partial U \cap B_\delta(x_0)$, $\geq \varphi(x)$
in $\partial U \setminus B_\delta(x_0)$ $\geq \varphi(x)$
$\Rightarrow \geq u_\varphi(x)$

$\Rightarrow |\varphi(x_0) - u_\varphi(x)| \leq \varepsilon - K Q_\varphi(x)$, as $x \to x_0$, $\varepsilon \to 0$. $\boxtimes$

Before we introduce the uniqueness of Poisson equation, we need to review some knowledge about sobolev spaces.

Firstly, for a region $U \subset \mathbb{R}^n$, function $u, v \in L^1_{local}(U)$.
We say v is a α-th weak derivative of u if

for all test functions $\varphi \in C_c^\infty(U)$, we have: $\int_U \varphi v = \int_U (-1)^{|\alpha|} u D^\alpha \varphi$

And we consider the space:

$$H^k(U) = \{\varphi \in L^1_{loc}(U),\ D^\alpha \varphi \subset L^2(U) \text{ for } |\alpha| \le k \text{ in weak sense}\}$$

And by this we can easily define the weak solution of a partial differential function.

$$\int_U (\Delta u + f)\varphi = 0 \overset{\text{weak sense}}{\Longleftrightarrow} \int_U f\varphi + u\Delta\varphi = 0 \quad \text{for all } \varphi \in C_c^\infty(U)$$

And we say u is a weak solution of $-\Delta u = f$ iff $\Uparrow$ holds

Meanwhile, we say $u_n \to u$ in $H^k(U)$, iff

$$\lim_{n\to\infty} \int_U \varphi(D^\alpha u_n - D^\alpha u) = 0 \text{ for all } |\alpha| \le k,\ \varphi \in C_c^\infty(U).$$

And we define the norm in $H^k(U)$ by

$$\|h\|^2_{H^k(U)} = \sum_{|\alpha| \le k} \|D^\alpha h\|^2_{L^2}$$

$H^k(U)$ forms a complete space under this norm.
And we define $H_0^k(U) = \overline{C_c^\infty(U)}$ under the norm.

We propose 2 common inequalities in $H^1(U)$ and $H_0^1(U)$

$\|fg\|_{L^1} \leq$

(Poincare. version 1) For $u \in H_0^1(U)$, where U is bounded in some direction we have: $\int_U u^2\,dx \leq C_{(U)} \int_U |\nabla u|^2\,dx$

Proof: Notice that: we only need to prove for C_c^∞ functions.

(Since $\forall u \in H_0^1(U)$, $\exists w_n \in C_c^\infty(U)$, s.t $\lim_{n\to\infty} \|u-w_n\|_{L^2(U)} = 0$, $\lim_{n\to\infty} \|\nabla u - \nabla w_n\|_{L^2(U)} = 0$
$\Rightarrow \lim_{n\to\infty} \|w_n\|_{L^2(U)} = \|u\|_{L^2(U)}$, $\lim_{n\to\infty} \|\nabla w_n\|_{L^2(U)} = \|\nabla u\|_{L^2(U)}$ ...).

And we may assume for simplicity where $\operatorname{spt} u \subset [0,L] \times \mathbb{R}^{n-1}$ (U's bdd direction.) and then extend it to $\mathbb{R}^n$

$$\Rightarrow |u(x_1, x_2, \dots, x_n)|^2 = |u(x_1, x_2, \dots, x_n) - u(0, x_2, \dots, x_n)|^2$$
$$= \left|\int_0^{x_1} u_{x_1}(s, x_2, \dots, x_n)\,ds\right|^2 \leq \left(\int_0^{x_1} 1^2\,ds\right)\left(\int_0^{x_1} |u_{x_1}(s, x_2, \dots, x_n)|^2\,ds\right)$$
$$\leq L \int_0^{x_1} |\nabla u(s, x_2, \dots, x_n)|^2\,ds$$

Integrating, ☒

(Poincare, version 2). For $u \in H^1(U)$, define $\bar{u} := \frac{1}{|U|}\int_U u(y)\,dy$
Then $\int_U |u-\bar{u}|^2\,dx \leq C_{(U)} \int_U |\nabla u|^2\,dx$

Pf: We only prove for 1-dim. $H^1(U) =$ absolutely cts functions with L^2 derivative
$U = (0,L)$, $\exists x_0$, s.t $u(x_0) = \bar{u}$

$$\Rightarrow |u(x) - u(x_0)|^2 = \left|\int_{x_0}^{x} \left(\text{/} \int_x^{x_0} \text{ doesn't matter}\right) u'(s)\,ds\right|^2 \leq L \cdot \int_{x_0}^{x} |u'(s)|^2\,ds \quad \dots ☒$$

Now, we can begin to think about another way to deduce the solution of Poisson equation

Consider the space. $X_g = \{g + C_0^\infty(U)\}$ and $I[u] = \int_U \frac{1}{2}|\nabla u|^2 - uf$.

① The minimizer of $I[u]$ in X_g corresponding to the solution of Poisson eq.

$\Rightarrow$: for any $\varphi \in C_c^2(U)$, $u+\varepsilon\varphi \in X_g$

Consider $i(\varepsilon) = I[u+\varepsilon\varphi]$, then $i'(0)$ should equal 0

$$\Rightarrow i'(0) = \int_U \nabla u \cdot \nabla \varphi - \varphi f = 0$$

$$= \int_U -\varphi \Delta u - \varphi f = 0 \Rightarrow -\Delta u = f.$$

$\Leftarrow$: If $-\Delta u = f$, consider $w \in C_c^2(U)$

We need to prove $\int_U \frac{1}{2}|\nabla u|^2 - uf \le \int_U \frac{1}{2}|\nabla u + \nabla w|^2 - (u+w)f$

$$\Leftrightarrow \int_U wf \le \int_U \frac{1}{2}|\nabla w|^2 + \nabla u \cdot \nabla w$$

$$LHS = \int_U -(\Delta u)w = \int_{\partial U} -\frac{\partial u}{\partial n} w \, dS + \int_U \nabla u \cdot \nabla w = \int_U \nabla u \cdot \nabla w$$ ☒

② If the minimizer exists, then its unique.

Proof: just consider $I[\frac{u_1+u_2}{2}] \le \frac{I[u_1]}{2} + \frac{I[u_2]}{2}$

In fact, $I[\frac{u_1+u_2}{2}] = \int_U \frac{1}{8}|\nabla u_1 + \nabla u_2|^2 - (\frac{u_1+u_2}{2})f$

$$I[\frac{u_1}{2}] + I[\frac{u_2}{2}] - I[\frac{u_1+u_2}{2}] \ge \int_U \frac{1}{8}|\nabla u_1 - \nabla u_2|^2 dx$$

If there exists 2 minimizer u_1, u_2, then $\frac{u_1+u_2}{2}$ is a minimizer as well.

$\Rightarrow$ we have: $\nabla u_1 \equiv \nabla u_2$, $u_1 = u_2$ in ∂U $\Rightarrow$ $u_1 \equiv u_2$

But until here, we didn't answer the whole question:

If there exists a weak solution ?

This corresponding to an important property.

(Weak compactness for $H^1(U)$):

If $\{U_n\}$ in $H^1(U)$ satisfying $\|U_n\|_{H^1(U)} \le M$ for all $n \in N^+$, then we have:

$\exists$ subsequence $\{U_{n_k}\} \rightharpoonup U_*$ in $H^1(U)$, i.e., $\forall v \in L^2(U)$

$$\int_U U_{n_k} V \to \int_U U_* V, \int_U \partial_{x_i} U_{n_k} V \to \int_U \partial_{x_i} U_* V$$

Now, we can consider the new space $X_g = g + H_0^1(U)$, $g \in C(\partial U)$, $f \in L^2(U)$

The above theorem still holds. since $H_0^1(U)$ forms zero boundary condition in the meaning of trace for Poisson equation.

Firstly, we prove the lower semi-continuous.

If $U_m \rightharpoonup U$ in H^1, then $\liminf_{m\to\infty} I[U_m] \geq I[U]$.

Proof: Firstly, since $f \in L^2(U) \Rightarrow \lim_{m\to\infty} \int_U U_m f = \int_U Uf$

And $\lim_{m\to\infty} \int |\nabla U_m|^2 - |\nabla U|^2 = \lim_{m\to\infty} \int_U |\nabla U_m - \nabla U|^2 - 2\overset{\in L^2\,!}{\nabla U}(\nabla U - \nabla U_m) \geq 0$ ☒

Now, consider the minimizer sequence of $I[U] \triangleq U_n$.

there exists M, s.t $I[U_n] \le M$ for all n.

Let $v_n = U_n - U_1$ in $H_0^1(U)$

$$I[U_n] = \int_U \frac{1}{2}|\nabla U_1 + \nabla V_n|^2 - f(U_1 + V_n) \le M$$

$$\geq \int_U \frac{1}{2}|\nabla U_1|^2 + \frac{1}{2}|\nabla V_n|^2 - \int_U \left(\frac{\varepsilon_1}{2}|\nabla V_n|^2 + \frac{1}{2\varepsilon_1}|\nabla U_1|^2\right) - \int_U \frac{fU_1}{2} - \int_U \left(\frac{f^2}{\varepsilon 2} + \frac{\varepsilon}{2} V_n^2\right)$$

$$\geq C + \left(\frac{1}{2} - \frac{\varepsilon_1}{2} - \frac{C_\varepsilon}{2}\right)\int_U |\nabla V_n|^2 \Rightarrow \int_U |\nabla V_n|^2 \le M' \Rightarrow \|V_n\|_{H_0^1(U)} \le M''$$

$\Rightarrow V_{n_k} \rightharpoonup V^* \text{ in } H^1 \Rightarrow U_{n_k} \rightharpoonup U^* \text{ in } H^1 \Rightarrow \liminf_{k\to+\infty} I[U_{n_k}] \geq I[U_*]$ ☒

(It's not hard to check minimizer in this space is a weak solution)

Now, we consider a different variation problem to solve

$\begin{cases} -\Delta u = f, U. \\ \frac{\partial u}{\partial n} = 0, \partial U \end{cases}$ ⊛ $I[u]$ is the same.

Then, we have:

① $\int_U f = 0$ for the solution of ⊛ $\Rightarrow I[u+c] = I[u], \forall c \in \mathbb{R}$

Pf: $\int_U -\Delta u \, dx = -\int_{\partial U} \frac{\partial u}{\partial n} dS = 0$

② u is a minimizer of $I[u]$ in $C^2(U) \cap C^1(\bar{U}) \Leftrightarrow u$ solves ⊛

Proof: $\Rightarrow \forall \varphi \in C_0^2(U)$, we have: (Assume that $u = \underset{C^2(U)\cap C(\bar{U})}{\arg\min}\, I[u]$

$i(\varepsilon) = I[u+\varepsilon\varphi]$, $i'(0) = 0$.

Consider $i'(\varepsilon) = \int_U \nabla u \cdot \nabla \varphi - \varphi f = \int_U -\varphi(f + \Delta u) + \int_{\partial U} \overset{=0}{\frac{\partial u}{\partial n}} \varphi$

$\Rightarrow f + \Delta u \equiv 0$, and take $\varphi \in C^2(U)$, we have $\frac{\partial u}{\partial n} = 0$ in ∂U.

$\Leftarrow$: Prove that: $\int_U \frac{1}{2}|\nabla u|^2 - uf \le \int_U \frac{1}{2}|\nabla w|^2 - wf$. for u solves ⊛ with $w \in C^2(U) \cap C(\bar{U})$

$\Leftrightarrow \int_U \frac{1}{2}(|\nabla u| + |\nabla w|)(|\nabla u| - |\nabla w|) \le \int_U (u-w) f$

Prove this may be hard. change, let $w = u + v$

$\Leftarrow \int_U vf \le \int_U \frac{1}{2}|\nabla v|^2 + \nabla u \cdot \nabla v$ $\quad \int_U \nabla u \cdot \nabla v = \int_{\partial U} \frac{\partial u}{\partial n} v \, dS - \int_U \Delta u v \ldots$ ∎

Similarly, Take minimizing sequence and using Poincaré for $H^1(U)$, we can assume $\bar{u} = \frac{1}{|U|}\int_U u \, dx = 0$, since $\pm$ constant doesn't change $I[u]$, then, we can select a weak limit as a minimizer of $I[u]$ in $H^1(U)$. ...

Before we end the review, we propose 2 stability result.

I) (L^2-stability) : $u \in C^2(U) \cap C(\bar{U})$ solve $\begin{cases} -\Delta u + cu = f, & U \\ u = 0, & \partial U \end{cases}$

with $c(x) \geq 0$ in U, $f \in L^2(U)$

If U is bdd, $\exists C$, s.t $\int_U |u|^2 + |\nabla u|^2 \leq C \int_U f^2$

If $c(x) \geq c_0 > 0 \Rightarrow \int_U |\nabla u|^2 + \frac{c_0}{2} \int_U |u|^2 \leq C \int_U f^2$

Proof: $-u\Delta u + cu^2 = uf$ in U

$\Rightarrow \int_U -u\Delta u + cu^2 dx = \int_U uf$

$\Rightarrow \int_U |\nabla u|^2 + c(x) u(x)^2 dx \leq \frac{\varepsilon}{2} \int_U u^2 + \frac{1}{2\varepsilon} \int_U f^2$

From Poincare, this is obviously

II) (L^∞ stability) : $u \in C^2(U) \cap C(\bar{U})$, $c \geq 0$.

$\begin{cases} -\Delta u + cu = f, & U \\ u = g, & \partial U \end{cases} \Rightarrow \max_{\bar{U}} |u| \leq C(\sup|f| + \sup|g|)$ (U is bdd)

Proof: This kind's proof is always related to some kind of maximum principle.

We prove if $\begin{cases} -\Delta u + cu \leq -\Delta v + cv, & U \\ u \leq v & \text{in } \partial U \end{cases} \Rightarrow u \leq v$ in $\bar{U}$

Only need to show $\begin{cases} -\Delta w + cw \leq 0, & U \\ w \leq 0 & \text{in } \partial U \end{cases} \Rightarrow w \leq 0$ in $\bar{U}$

Assume the contrary, if there exists $w > 0$ in $\bar{U}$, then the maximum point must in U, let $x_0 = \arg\max_{\bar{U}} w(x)$

$\Rightarrow cw \geq 0, -\Delta w \geq 0 \Rightarrow -\Delta w + cw \geq 0$

If $\Delta w + cw < 0$. the theorem holds, since contradiction appears.

If $\Delta w + cw \leq 0$, consider, $w_\varepsilon + \varepsilon(e^{x_1} - L)$, $L > e^{x_1}$, when, $x \in U$.

$w_\varepsilon \rightrightarrows w$, which finishes the proof.

Back to our proof. Let $F = \sup|f|$, $G = \sup|g|$.

$G + F(L - e^{x_1}M) = v(x)$. $-\Delta v(x) = FMe^{x_1}$, choose M, s.t $Me^{x_1} > 2, \forall x$

Choose L, s.t $L > Me^{x_1}$, $\forall x$ $\blacksquare$